Intrinsic dimensionality

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Scope of the lecture

- Understand a basic problem of multidimensional signal processing.
- Learn a way of solving this problem by using a combination of differential geometry and filter theory.



Plan of the lecture

- Introduction and motivation
- Definition of the problem
- Images as surfaces, introduction to differential geometry and curvature
- Compensation principle
- Volterra-Wiener systems
- Reconstruction



Intrinsic dimensionality in 2d

A signal $f(x,y)$ has 2 degrees of freedom but does not always use them.

- ***i0D*** signals: constant in all directions

$$f(x,y) = const.$$

Fourier Transform

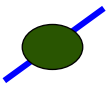
- ***i1D*** signals: constant in one direction

$$f(x,y) = g(\xi)$$



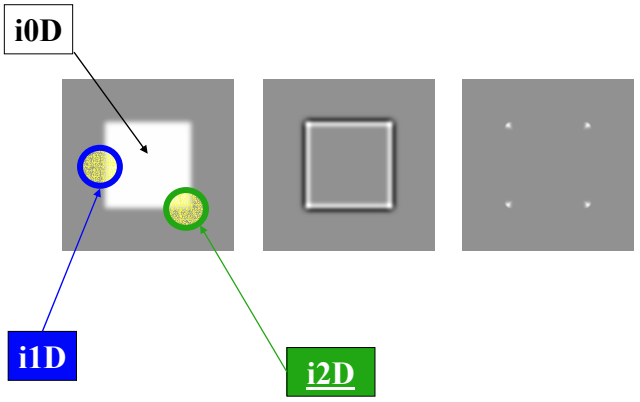
- ***i2D*** signals: no constant direction

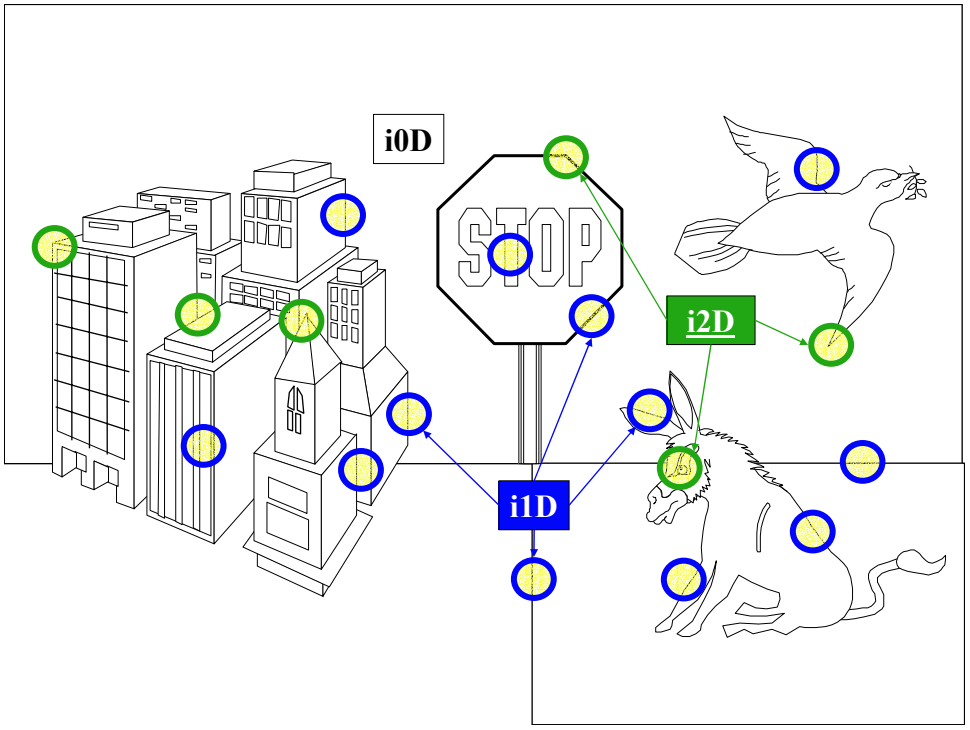
$$f(x,y) = g(\xi,\psi)$$





inD examples

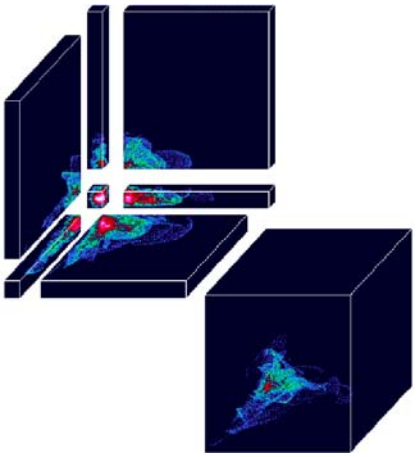




Motivation: statistics of natural images

The figure shows the distribution of oriented-filter outputs for a large number of natural images:
• i0D signals do not activate any filter and are therefore located at the origin (the small cube)
• i1D signals that activate only one orientation are located on an axis
• i2D signals that activate more than one filter are located in the volume (large cube).

Note that i2D signals are rare in natural images.



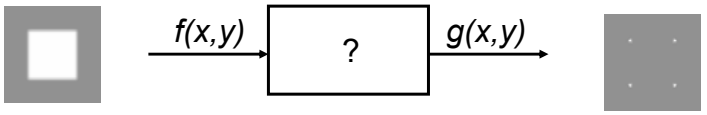


Motivation: properties of *inD* signals

- *i0D* and *i1D* signals are redundant
- *i2D* signals are rare in natural images
- *i2D* representations are sparse
- *i2D* representations can be unique
- *i2D* representations are efficient
- motion can only be estimated for *i2D* signals
- *i2D* signals are good features to track
- saccades more often land at *i2D* signals
- endstopped neurons respond to *i2D* signals only



Definition of the problem: i2D systems



Desired output: **if f is i0D -> g=0**

if f is i1D -> g=0

and, of course,

if f is i2D -> $g \neq 0$

An *i2D system* blocks all *i0D* and *i1D* signals.
An *i2D* system is also called *i2D operator*.



The problem in the Fourier domain

An *i2D system* blocks signals that have their Fourier transform restricted to a Dirac-delta line through the origin.
In analogy, a high-pass filter blocks signals that have their Fourier transform restricted to a Dirac-delta at origin. For one-dimensional signals, this is the only possible Dirac-delta distribution.
Multi-dimensional signals can have their spectral energy restricted to lines, planes, hyperplanes.
Note, however, that this is not a quantitative issue!
Therefore, … next slide



A recommendation

Please do not think about multi-dimensional signals from the perspective of one-dimensional signals.

Be skeptical if someone tells you that a theory developed for one dimension would be good for more dimensions, or anytime someone says “and obviously this result/theory can be extended from one to more dimensions”.

The problem of distinguishing between i1D and i2D signals will not appear with one-dimensional signals and will therefore not be solved with theories extended from one dimension.



Main difficulty with i2D systems

An *i2D system* cannot be linear, because the eigenfunctions of linear systems are *i1D* signals (sinusoidal gratings).

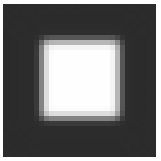
Nonlinear systems are more difficult to treat.

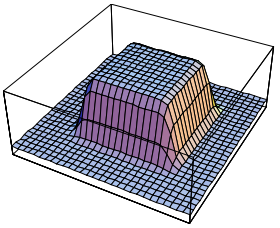


The geometric view: images as surfaces

Image intensity $f(x,y)$

Image surface $\mathbf{S} = (x, y, f(x,y))$

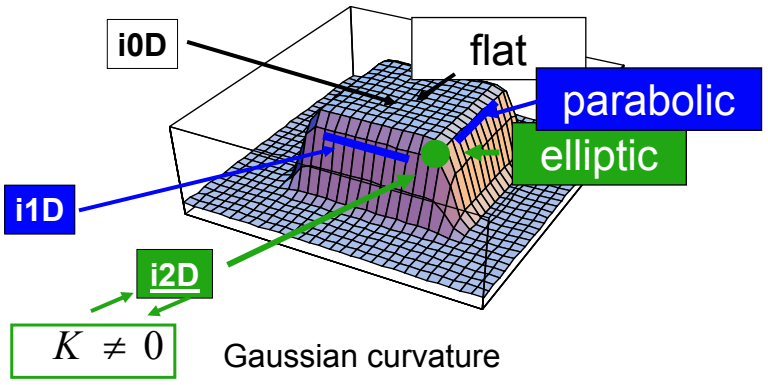




Note: Surface geometry is also useful in CV for computing invariant properties of object surfaces from range images.

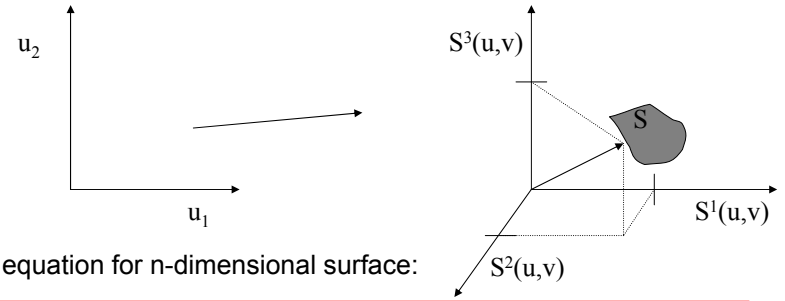


Intrinsic dimension and curvature





Surface curvature: definition



Parametric equation for n-dimensional surface:

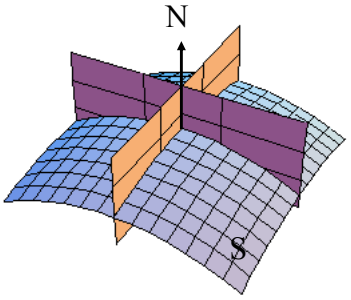
$$\mathbf{S}(\mathbf{u}) = (S^1(\mathbf{u}), S^2(\mathbf{u}), \ldots, S^{n+1}(\mathbf{u})); \quad \mathbf{u} = (u_1, u_2, \ldots, u_n)$$

The definition of Gaussian curvature is:

$$K = \lim_{\Delta A \to 0} \frac{\Delta \Omega}{\Delta A}$$

where is a portion of S and $\Delta\Omega$ is the spherical image of ΔA.

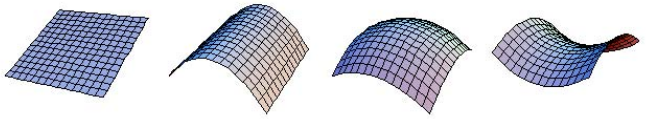
Surface curvature: main curvatures



$$K = \prod_i k_i$$

with k_i being the main curvatures.

Surface types: planar, parabolic, elliptic, and hyperbolic



$K = 0$ $K = 0$ $K > 0$ $K < 0$

Surface curvature: differential results

The first-fundamental-form coefficients are

$$g_{ij} = \mathbf{S}_i \mathbf{S}_j; \quad \mathbf{S}_{ij} = \frac{\partial^2 \mathbf{S}}{\partial u^i \partial u^j}$$

and the second-fundamental-form coefficients

$$b_{ij} = \mathbf{S}_{ij} \mathbf{N}; \quad \mathbf{N} = \frac{\mathbf{S}_1 \times \mathbf{S}_1 \times \ldots \times \mathbf{S}_n}{\left| \mathbf{S}_1 \times \mathbf{S}_1 \times \ldots \times \mathbf{S}_n \right|}$$

Based on the fundamental-form coefficients, Gaussian curvature can be computed, for an image surface, as

$$\mathbf{S} = (x, y, f(x,y)) \longrightarrow K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$



Determinant and eigenvalues of the Hessian

The numerator D of K determines the sign of K

$$D = f_{xx} f_{yy} - f_{xy}^2$$

D is the determinant of the Hessian matrix

$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$D = \lambda_1 \lambda_2$ λ_1, λ_2 Eigenvalues of *H*, therefore

$E = Min(\lambda_1, \lambda_2)$ is a 2D operator



Intrinsic dimensionality and structure tensor

The intrinsic dimensionality can be determined by using the structure tensor J

$$J = g * \begin{vmatrix} f_x^2 & f_x f_y \\ f_x f_y & f_y^2 \end{vmatrix}$$

The intrinsic dimension is related to the rank of J

$\lambda_1 = \lambda_2 = 0$ rank 0, *i0D*

$\lambda_{\min} = 0$ rank 1, *i1D*

$\lambda_1, \lambda_2 \neq 0$ rank 2, *i2D*

λ_1, λ_2 are the Eigenvalues of J.



Computation of derivatives

- Fourier correspondence:

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} f \quad \xleftrightarrow{FT} \quad (j\omega_x)^n (j\omega_y)^m F; \quad with\ f \xleftrightarrow{FT} F$$

- Blurred derivatives:

$\frac{\partial^{n+m}}{\partial x^n \partial y^m} f$ replaced with $(\frac{\partial^{n+m}}{\partial x^n \partial y^m} g) * f$

where $g = \exp(-\frac{\pi(x^2 + y^2)}{\sigma^2})$ and * denotes convolution



i2D operators: compensation principle

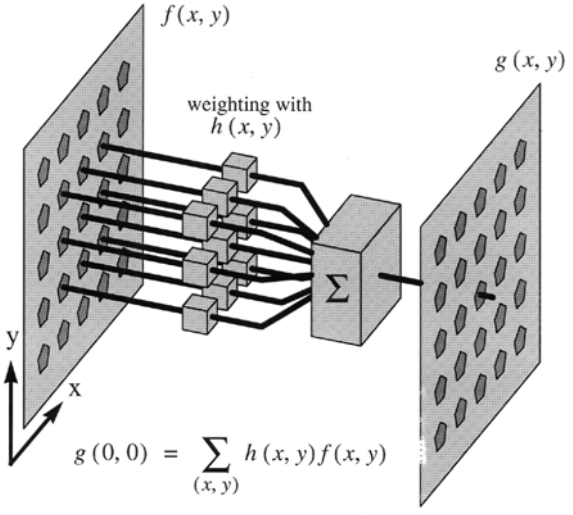
Determinant of the Hessian

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} \longrightarrow D = (g_{xx} * f)(g_{yy} * f) - (g_{xy} * f)^2$$

A class of 2D operators: $A = P - C$

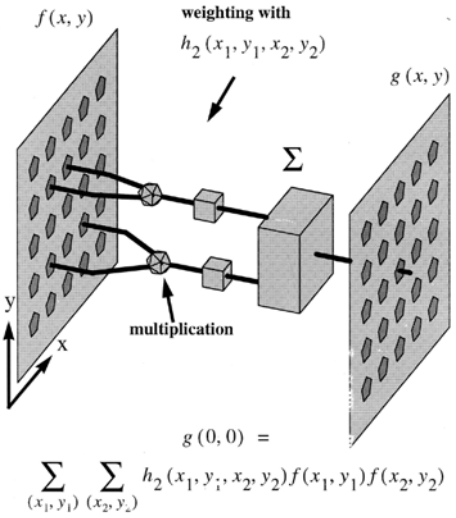
- Recall that if the input (f) is *i1D*, the Fourier Transform (F) is restricted to a line and the output of a *i2D* operator must be zero, and P must equal C.
- If P and C where linear filters, A would always be zero.
- P must generate product terms, which lead to convolution integrals in the Fourier Transform.
- Compensation principle: all one-dimensional convolution integrals in the Fourier transform of A must cancel! This rule can be used to design C.

Linear systems



Output values are weighted sums of input values.

Volterra systems

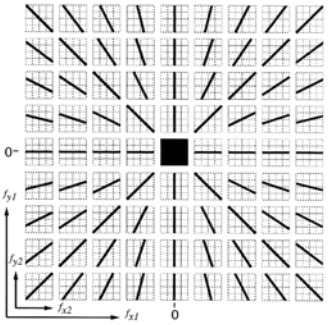


A 2D quadratic nonlinear system can be described as a 4D linear system. The output values result from pairs of input values that are multiplied, then weighted and summed.

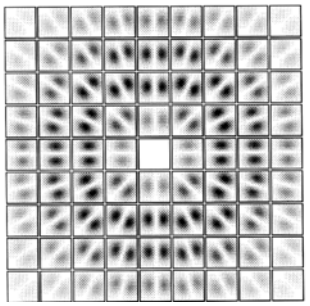
i2D operators as Volterra systems

To be a i2D operator a Volterra system must satisfy (H is the Fourier transform of the 2nd-order Volterra kernel; f_{x1}, ... are the frequencies):

$$H(f_{x1}, f_{y1}, f_{x2}, f_{y2}) = 0 \quad \forall \; f_{x1} f_{y2} = f_{y1} f_{x2}$$

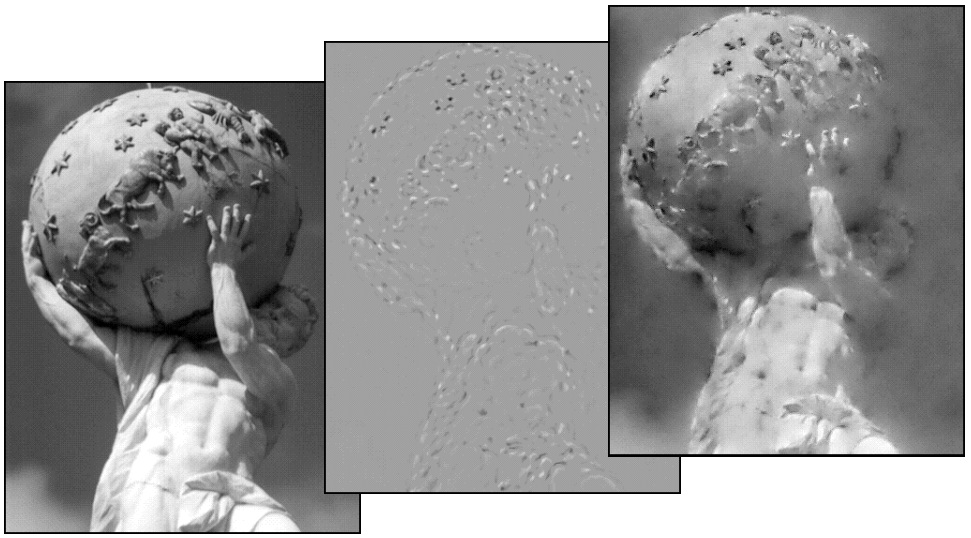


Forbidden zones

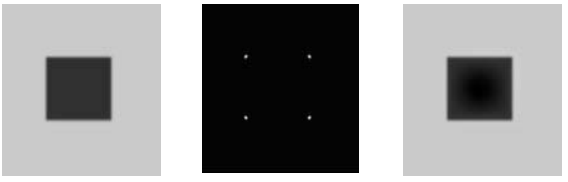


Spectrum of D operator

Reconstruction from i2D features



Reconstruction of a square from its corners



Original image

Reconstructed image

$$f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}$$

Result due to CICERO MOTA.

Summary

- The intrinsic dimension is an important and basic property of a multidimensional signal
- *i2D* systems cannot be treated with linear theories.
- Differential geometry is useful for describing *i2D* systems.
- The compensation principle states that nonlinearities must generate product terms and that all one-dimensional convolution integrals of the product terms must cancel.
- A special class of *i2D* systems can be described as Volterra-Wiener systems.
- *i2D* features can be unique, therefore i1D features are redundant.

